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- 1) Consider the two-dimensional system in $\mathbb{R}^2$

$$\begin{aligned}\dot{x} &= y(1 + x - y^2) \\ \dot{y} &= x(1 + y - x^2)\end{aligned}$$

Determine the critical points and characterise the linearised flow in a neighbourhood of these points.

- 2) Consider the system

$$\begin{aligned}\dot{x} &= 16x^2 + 9y^2 - 25 \\ \dot{y} &= 16x^2 - 16y^2\end{aligned}$$

- a) Determine the critical points and characterise them by linearisation.
- b) Sketch the phase-flow.

- 3) In certain applications one studies the equation

$$\ddot{x} + c\dot{x} - x(1 - x) = 0$$

with a special interest in solutions with the properties:

$$\lim_{t \rightarrow -\infty} x(t) = 0, \quad \lim_{t \rightarrow \infty} x(t) = 1, \quad \dot{x}(t) > 0 \quad \text{for } -\infty < t < \infty.$$

Derive a necessary condition for the parameter c for such solutions to exist.

- 4) Find the critical points of the system

$$\begin{aligned}\dot{x} &= x(1 - x^2 - 6y^2) \\ \dot{y} &= y(1 - 3x^2 - 3y^2)\end{aligned}$$

Characterise the critical points by linear analysis.

- 5) Find the stable, unstable and center subspaces E^s , E^u and E^c of the linear systems with the matrix

a) $A = \begin{pmatrix} -1 & -3 \\ 0 & 2 \end{pmatrix}$

b) $A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$

c) $A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Also, sketch the phase portrait in each of these cases. Which of these matrices define a hyperbolic flow?

- 6) Solve the linear system $\dot{x} = Ax$ with

$$A = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 6 \end{pmatrix}$$

Find the stable, unstable and center subspaces E^s , E^u and E^c for this system and sketch the phase portrait. For $x_0 \in E^c$, show that the sequence of points $x_n = e^{An}x_0 \in E^c$; similarly, for $x_0 \in E^s$ or E^u , show that $x_n \in E^s$ or E^u respectively.

- 7) Classify the equilibrium points of the Lorenz equation with

$$\begin{aligned}\dot{x} &= y - x \\ \dot{y} &= \mu x - y - xz \\ \dot{z} &= xy - z\end{aligned}$$

for $\mu > 0$. At what value of the parameter μ do two new equilibrium points “bifurcate” (a change in the number of equilibria) from the equilibrium point at the origin? Hint: For $\mu > 1$, the eigenvalues at the nonzero equilibrium points are $\lambda = -2$ and $\lambda = (1 \pm \sqrt{5 - 4\mu})/2$.

- 8) Find the approximations of the stable W_{LOC}^S and unstable W_{LOC}^U local manifolds near the origin using the Taylor method (of order 2, 3 and 4) for the system:

$$\begin{aligned}\dot{x}_1 &= -x_1 \\ \dot{x}_2 &= x_2 + x_1^2\end{aligned}$$

Can we say something about W^S and W^U ? Solve the system and show that W^S and W^U are given by $W^S : x_2 = -x_1^2/3$ and $W^U : x_1 = 0$. Sketch E^S , E^U , W^S and W^U .