

- 1) Use Dulac's criteria to prove that the system

$$\begin{aligned}\dot{x} &= -y + x(1 - 2x^2 - 3y^2) \\ \dot{y} &= x + y(1 - 2x^2 - 3y^2)\end{aligned}$$

has a unique limit cycle in an annulus. (Hint: use polar coordinates)

- 2) Prove that none of the following systems have any limit cycles:

a) $\dot{x} = 1 + y^2 - e^{xy}, \quad \dot{y} = xy + \cos^2 y.$

b) $\dot{x} = y^2 - x, \quad \dot{y} = y + x^2 + yx^3.$

c) $\dot{x} = y + x^3, \quad \dot{y} = x + y + y^3.$

d) $\dot{x} = 2xy - 2y^4, \quad \dot{y} = x^2 - y^2 - xy^3.$

e) $\dot{x} = x(2 - y - x), \quad \dot{y} = y(4x - x^2 - 3)$ (Hint: use $B(x, y) = 1/(xy)$).

- 3) Prove that the system

$$\dot{x} = x(1 - 4x + y), \quad \dot{y} = y(2 + 3x - 2y)$$

has no limit cycles by applying Bendixson's criteria with $B(x, y) = x^m y^n$

- 4) In the system $\dot{x} = f(y), \quad \dot{y} = g(x) + y^k$ ($k \in \mathbb{N}$), the functions f and g are $\mathcal{C}^1$ functions.
- a) Give sufficient conditions for k so that the system contains no periodic solutions.
- b) Choose $f(y) = -y, \quad g(x) = x, \quad k = 2$. Does the system contain periodic solutions, cycles or limit cycles?

- 5) Consider the system

$$\begin{aligned}\dot{x} &= y(1 + x - y^2) \\ \dot{y} &= x(1 + y - x^2)\end{aligned}$$

but suppose that the equations model an experimental situation such that $x \geq 0, y \geq 0$ (for instance because x and y are quantities in chemical reactions). Do periodic solutions exist?

- 6) In $\mathbb{R}^n$ we consider the equation $\dot{x} = f(x)$ and a point a ; $f(x)$ is continuously differentiable. Suppose that a solution $\varphi(t)$ of the equation exists such that

$$\lim_{t \rightarrow \infty} \varphi(t) = a.$$

Show that a is a critical point of the equation.

7) Given

$$\begin{aligned}\dot{x} &= -y + x(1 - x^2 - y^2) \sin(1 - x^2 - y^2)^{-1/2} \\ \dot{y} &= x + y(1 - x^2 - y^2) \sin(1 - x^2 - y^2)^{-1/2}\end{aligned}$$

for $x^2 + y^2 \neq 1$ with $\dot{x} = -y$ and $\dot{y} = x$ for $x^2 + y^2 = 1$. Show that there is a sequence of limit cycles

$$\Gamma_n^\pm : r = \sqrt{1 \pm \frac{1}{n^2 \pi^2}}$$

for $n = 1, 2, 3, \dots$ which accumulates on the cycle

$$\Gamma : \mathbf{x} = \gamma(t) = (\cos t, \sin t)^T.$$

What can you say about the stability of $\Gamma_n^\pm$?

8) Consider the system

$$\begin{aligned}\dot{x} &= -y + x(r^4 - 3r^2 + 1) \\ \dot{y} &= x + y(r^4 - 3r^2 + 1)\end{aligned}$$

where $r^2 = x^2 + y^2$.

- Show that $\dot{r} < 0$ on the circle $r = 1$ and that $\dot{r} > 0$ on the circle $r = 2$. Use the Poincare-Bendixson Theorem and the fact that the only critical point of this system is at the origin to show that there is a periodic orbit in the annular region $A_1 = \{\mathbf{x} \in \mathbb{R}^2 \mid 1 \leq |\mathbf{x}| \leq 2\}$.
- Show that the origin is an unstable focus for this system and use the Poincare-Bendixson Theorem to show that there is a periodic orbit in the region $A_2 = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| \leq 1\}$.
- Find the unstable and stable limit cycles of this system.

9) Use the Poincare-Bendixson Theorem and the fact that the planar system

$$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3$$

has only the one critical point at the origin to show that this system has a periodic orbit in the annular region $A = \{\mathbf{x} \in \mathbb{R}^2 \mid 1 < |\mathbf{x}| < \sqrt{2}\}$. Hint: Convert to polar coordinates and show that for all $\varepsilon > 0$, $\dot{r} < 0$ on the circle $r = \sqrt{2} + \varepsilon$ and $\dot{r} > 0$ on $r = 1 - \varepsilon$; then use the Poincare-Bendixson theorem to show that this implies that there is a limit cycle in

$$\bar{A} = \{\mathbf{x} \in \mathbb{R}^2 \mid 1 \leq |\mathbf{x}| \leq \sqrt{2}\}$$

and then show that no limit cycle can have a point in common with either one of the circles $r = 1$ or $r = \sqrt{2}$. Show that there is at least one stable limit cycle in A .

10) Show that the system

$$\dot{x} = \frac{y}{1 + x^2}, \quad \dot{y} = \frac{-x + y(1 + x^2 + x^4)}{1 + x^2}$$

has no limit cycle in $\mathbb{R}^2$.