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1) Determine the stability using the Lyapunov theory of the origin of the system $\dot{\mathbf{x}} = f(\mathbf{x})$ with $f(\mathbf{x})$ given by

   a) $\dot{x} = -x + y - xy^2, \qquad \dot{y} = -2x - y - x^2 y.$

   b) $\dot{x} = y^3 + x^2 y, \qquad \dot{y} = x^3 - xy^2.$

   c) $\dot{x} = -x^3 + y^4, \qquad \dot{y} = -y^3 + y^4.$

2) Show that the zero solution of

$$\dot{x} = y + xy^2, \qquad \dot{y} = x + x^2 y$$

   is unstable.

3) Show that the origin of the system

$$\dot{x} = -y - x\sqrt{x^2 + y^2}, \qquad \dot{y} = x - y\sqrt{x^2 + y^2}$$

   is a centre in the linear approximation, but in fact is a stable spiral. Find a Lyapunov function for the zero solution.

4) The origin is a center for the linear system

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}$$

   The addition of nonlinear terms to the right-hand side of this linear system changes the stability of the origin. Use the Lyapunov function $V(\mathbf{x}) = x_1^2 + x_2^2$ to establish the following results:

   a) The origin is an asymptotically stable equilibrium point of

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -x_1^3 - x_1 x_2^2 \\ -x_2^3 - x_2 x_1^2 \end{pmatrix}$$

   b) The origin is an unstable equilibrium point of

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} x_1^3 + x_1 x_2^2 \\ x_2^3 + x_2 x_1^2 \end{pmatrix}$$

   c) The origin is a stable equilibrium point which is not asymptotically stable for

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -x_1 x_2 \\ x_1^2 \end{pmatrix}$$

   What are the solution curves in this case?

5) Use the Liapunov function $V(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2$ to show that the origin is an asymptotically stable equilibrium point of the system

$$\begin{aligned} \dot{x_1} &= -x_2 - x_1 x_2^2 + x_3^2 - x_1^3 \\ \dot{x_2} &= x_1 + x_3^3 - x_2^3 \\ \dot{x_3} &= -x_1 x_3 - x_3 x_1^2 - x_2 x_3^3 - x_3^5 \end{aligned}$$

Show that the trajectories of the linearized system $\mathbf{x} = Df(0)\mathbf{x}$ for this problem lie on circles in planes parallel to the (x_1, x_2) plane; hence, the origin is stable, but not asymptotically stable for the linearized system.

6) Consider the two-dimensional system

$$\begin{aligned} \dot{x} &= -y + f(x, y) \\ \dot{y} &= \sin x \end{aligned}$$

The function $f(x, y)$ is smooth and nonlinear. Give sufficient conditions for $f(x, y)$ so that $(0, 0)$ is a stable equilibrium solution. *(Hint: Inspired in the case $f \equiv 0$ use $V(x, y) = \frac{1}{2}y^2 - \cos x + K$, for a carefully chosen constant K).*

7) Consider the system

$$\begin{aligned} \dot{x} &= 2y(z - 1) \\ \dot{y} &= -x(z - 1) \\ \dot{z} &= xy \end{aligned}$$

a) Show that the solution $(0, 0, 0)$ is stable. *(Hint: Use $V(x, y) = I_1 + (I_2 - 1)^2$, with $I_1 = x^2 + 2y^2$ (integral using the first two equations) and $I_2 = y^2 + (z-1)^2$ (integral using the last two equations)).*

b) Is this solution asymptotically stable?

8) Determine the stability of the solution $(x, y) = (0, 0)$ of the system $\dot{x} = 2xy + x^3$, $\dot{y} = x^2 - y^5$. *(Hint: Use $V(x, y) = x^2 - Ky^2$, for a carefully chosen constant K).*

9) Determine the stability of the trivial solution of

$$\dot{x} = xy^2 - \frac{1}{2}x^3, \qquad \dot{y} = -\frac{1}{5}y^3 + \frac{1}{2}x^2 y$$

(Hint: Use $V(x, y) = ax^2 + by^2$, for carefully chosen constants a, b).