

1) Consider the system

$$\dot{x} = -y + \mu x + xy^2, \quad \dot{y} = x + \mu y - x^2$$

- a) (*Linearization*) By calculating the linearization at the origin, show that the system undergoes a Hopf bifurcation when  $\mu = 0$ .
- b) (*A computer analysis*) (**Se hará en prácticas**) By plotting phase portraits on the computer, show that the system undergoes a Hopf bifurcation when  $\mu = 0$ . Is it supercritical, subcritical, or degenerate?
- c) (*A heuristic analysis*)
  - i) Rewrite the system in polar coordinates.
  - ii) Show that if  $r \ll 1$ , then  $\dot{\theta} \approx 1$  and  $\dot{r} \approx \mu r + \frac{1}{8}r^3$ , where the terms omitted are oscillatory and have essentially zero time-average around one cycle.
  - iii) The above formulas suggest the presence of an unstable limit cycle of radius  $r \approx \sqrt{-8\mu}$  for  $\mu < 0$ . Confirm that prediction numerically. (Since we assumed that  $r \ll 1$ , the prediction is expected to hold only if  $|\mu| \ll 1$ .)
- d) (*Analytical criterion to decide if a Hopf bifurcation is subcritical or supercritical*) Use the criterion below to decide which type of Hopf bifurcation occurs at  $\mu = 0$ .

**(Analytical criterion)** Any system at a Hopf bifurcation can be put into the following form by suitable changes of variables

$$\dot{x} = -\omega y + f(x, y), \quad \dot{y} = \omega x + g(x, y)$$

where  $f$  and  $g$  contain only higher-order nonlinear terms that vanish at the origin. As shown by Guckenheimer and Holmes (1983), one can decide whether the bifurcation is subcritical or supercritical by calculating the sign of the following quantity ( $l_1$ , the first Lyapunov coefficient):

$$16l_1 = f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} \\ + \frac{1}{\omega} [f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}]$$

where the subscripts denote partial derivatives evaluated at  $(0, 0)$ . The criterion is: If  $l_1 < 0$ , the bifurcation is supercritical; if  $l_1 > 0$ , the bifurcation is subcritical.

- 2) Use the above criterion to decide if the Hopf bifurcation at origin for  $\mu = 0$  is subcritical or supercritical for the systems
  - a)  $\dot{x} = y + \mu x, \quad \dot{y} = -x + \mu y - x^2 y$
  - b)  $\dot{x} = \mu x + y - x^3, \quad \dot{y} = -x + \mu y + 2y^3$ .

- 3) Consider the biased van der Pol oscillator

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$$

Find the curves in  $(\mu, a)$  space at which Hopf bifurcation occur.

- 4) The Belgian scientists Prigogine and Lefever formulated a model for certain chemical reactions; the model is called the Brusselator. If there is no spatial diffusion the equations are

$$\begin{aligned}\dot{x} &= a - (b + 1)x + x^2y \\ \dot{y} &= bx - x^2y\end{aligned}$$

in which  $x$  and  $y$  are concentrations ( $x, y > 0$ ),  $a$  and  $b$  are positive parameters. Does the Brusselator admit the possibility of Hopf bifurcation?

- 5) Consider the system

$$\dot{x} = y - 2x, \quad \dot{y} = \mu + x^2 - y$$

Find and classify the bifurcations that occur as  $\mu$  varies. Sketch the phase portrait as a function of  $\mu$ .

- 6) In a study of isothermal autocatalytic reactions, Gray and Scott (1985) considered a hypothetical reaction whose kinematics are given in dimensionless form by

$$\dot{u} = a(1 - u) - uv^2, \quad \dot{v} = uv^2 - (a + k)v$$

where  $a, k > 0$  are parameters. Show that saddle-node bifurcations occur when the values of the parameters satisfy  $k = -a \pm \frac{1}{2}\sqrt{a}$ .